Two dimensional anelastic model deepconv-mars: Part II, Finite difference equations of the model

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Contents

1 Outline of discretizatoin

The outline of finite difference method adapted for this model is as follows.

Space differencing The finite difference form of governing equations of the model are considered on the Lorenz type staggered grid. The space differencing is evaluated by the forth order centered scheme for scalar advection terms and the continuity equation, and the second order centered scheme for others. The numerical diffusion is introduced to the equation of motion, turbulent kinetic energy equation, and advection diffusion equation of dust so that the 2-grid noise associated with central finite differencing can be suppressed. The numerical diffusion in equation of motion is proportional to the squared wind shear and that in turbulent kinetic energy equation, and advection diffusion equation of dust is proportional to the third power of Laplace operator.

The vertical integral in calculating CO_2 infrared radiative flux is evaluated by the trapezoidal rule.

Time differencing The time integration is performed by the leap-frog scheme for advection and buoyancy terms and the forward scheme for turbulent diffusion and forcing terms. The forward scheme is also adapted once per 20 steps for advection and buoyancy terms to stabilized numerical solution. The radiative flux associated with dust is given by iteration method of the matrix equation, where the number of iteration is 4. The time integration of 1D thermal conduction equation of grand surface is performed by the Crank-Nicolson scheme.

In the following sections, the subscripts i, j show horizontal and vertical grid point, and the superscripts n, N show time step. The number of vertical grid level is J. The scalar and basic state variables are evaluated on the grid point and the other variables are evaluated on the half grid point. Δx and Δz_j are the horizontal and vertical grid intervals, and Δt is the time interval.

2 Atmospheric model

2.1 Equation of motion

Before making finite difference equations, equation (1)∼(3) show in Part I are transformed as follows.

$$
\begin{aligned}
\frac{\partial u}{\partial t} &= -\frac{\partial \hat{P}}{\partial x} + \alpha, \\
\frac{\partial v}{\partial t} &= \beta, \\
\frac{\partial w}{\partial t} &= -\frac{\partial \hat{P}}{\partial z}
\end{aligned}
$$

where,

$$
\begin{aligned}
\hat{P} &\equiv c_p \Theta_0 \pi - \gamma, \\
\alpha &\equiv -u\frac{\partial u}{\partial x} - w\frac{\partial u}{\partial z} - fv + D(u) - \frac{\partial \beta}{\partial x}, \\
\beta &\equiv -u\frac{\partial v}{\partial x} - w\frac{\partial v}{\partial z} + fu + D(v), \\
\gamma &\equiv \int_0^z \left(-u\frac{\partial w}{\partial x} - w\frac{\partial w}{\partial z} + g\frac{\theta}{\Theta_0} + D(w) \right) dz
\end{aligned}
$$

In this formulation, the time dependence of upper and lower boundary conditions are disappeared.

The advection terms D[UVW]ADV are evaluated by the combination scheme of flux and advection forms. The time integration is performed by the forward scheme for the friction term $[\mathrm{D[UVW]VIS}]^N_{i+\frac{1}{2},j}$, $[\mathrm{D[UVW]NLV}]^N_{i+\frac{1}{2},j}$, and combination of the leap-frog and forward scheme for the other terms. The calculation method of pressure term $\hat{P}$ are shown in Section 2.3.

$$
\begin{aligned}
u^{n+1}_{i+\frac{1}{2},j} &= u^N_{i+\frac{1}{2},j} + dt\left\{ \frac{\hat{P}_{i+1,j} - \hat{P}_{i,j}}{\Delta x} + \alpha_{i+\frac{1}{2},j} \right\}, &(1)\\
v^{n+1}_{i+\frac{1}{2},j} &= v^N_{i+\frac{1}{2},j} + dt\beta^n_{i+\frac{1}{2},j}, &(2)\\
w^{n+1}_{i,j+\frac{1}{2}} &= w^N_{i,j+\frac{1}{2}} + dt\frac{\hat{P}_{i,j+1} - \hat{P}_{i,j}}{\Delta z_{j+\frac{1}{2}}}. &(3)
\end{aligned}
$$

$$N = \begin{cases} n-1 & \text{for} \quad \text{leap frog}, \\ n & \text{for} \quad \text{forward}, \end{cases} \qquad dt = \begin{cases} 2\Delta t & \text{for} \quad \text{leap frog}, \\ \Delta t & \text{for} \quad \text{forward}. \end{cases} \tag{4}$$

$$\begin{aligned}
\alpha_{i+\frac{1}{2},j} &= -(\gamma_{i+1,j} - \gamma_{i,j}) + [\mathrm{DUADV}]^n_{i+\frac{1}{2},j} + [\mathrm{DUPRS}]^n_{i+\frac{1}{2},j} + [\mathrm{DUCOLI}]^n_{i+\frac{1}{2},j} \\
&\quad + [\mathrm{DUVIS}]^N_{i+\frac{1}{2},j} + [\mathrm{DUNLV}]^N_{i+\frac{1}{2},j}, && (5) \\
\beta^n_{i+\frac{1}{2},j} &= [\mathrm{DVADV}]^n_{i+\frac{1}{2},j} + [\mathrm{DVCOLI}]^n_{i+\frac{1}{2},j} + [\mathrm{DVVIS}]^N_{i+\frac{1}{2},j} + [\mathrm{DVNLV}]^N_{i+\frac{1}{2},j}, && (6) \\
\gamma_{i,j} &= \sum_{j'=0}^{j} \Big([\mathrm{DWADV}]^n_{i,j'-\frac{1}{2}} + [\mathrm{DWVIS}]^N_{i,j'-\frac{1}{2}} + [\mathrm{DWNLV}]^N_{i,j'-\frac{1}{2}} \\
&\quad + [\mathrm{BUOY}]^n_{i,j'-\frac{1}{2}} \Big) \Delta z_{j'-\frac{1}{2}} && (7)
\end{aligned}$$

$$\begin{aligned}
\mathrm{DUADV}^n_{i+\frac{1}{2},j} &= -\frac{1}{\Delta x} \Big\{ \Big(u^n_{i+1,j} u^n_{i+1,j} - u^n_{i-1,j} u^n_{i-1,j} \Big) \\
&\quad + \Big(\rho_{0,j+\frac{1}{2}} u^n_{i+\frac{1}{2},j+\frac{1}{2}} w^n_{i+\frac{1}{2},j+\frac{1}{2}} - \rho_{0,j-\frac{1}{2}} u^n_{i+\frac{1}{2},j-\frac{1}{2}} w^n_{i+\frac{1}{2},j-\frac{1}{2}} \Big) \Delta x \Big/ (\rho_{0,j} \Delta z_j) \\
&\quad - u^n_{i+\frac{1}{2},j} \left(\nabla \cdot \rho_0 \boldsymbol{v} / \rho_0 \right)^n_{i+\frac{1}{2},j} \Big\}, && (8) \\
\mathrm{DVADV}^n_{i+\frac{1}{2},j} &= -\Big\{ \Big(v^n_{i+1,j} u^n_{i+1,j} - v^n_{i-1,j} u^n_{i-1,j} \Big) \\
&\quad + \Big(\rho_{0,j+\frac{1}{2}} v^n_{i+\frac{1}{2},j+\frac{1}{2}} w^n_{i+\frac{1}{2},j+\frac{1}{2}} - \rho_{0,j-\frac{1}{2}} v^n_{i+\frac{1}{2},j-\frac{1}{2}} w^n_{i+\frac{1}{2},j-\frac{1}{2}} \Big) \Delta x \Big/ (\rho_{0,j} \Delta z_j) \\
&\quad - v^n_{i+\frac{1}{2},j} \left(\nabla \cdot \rho_0 \boldsymbol{v} / \rho_0 \right)^n_{i+\frac{1}{2},j} \Big\} \Big/ \Delta x, && (9) \\
\mathrm{DWADV}^n_{i,j+\frac{1}{2}} &= -\frac{1}{\Delta x} \Big\{ \Big(w^n_{i+\frac{1}{2},j+\frac{1}{2}} u^n_{i+\frac{1}{2},j+\frac{1}{2}} - w^n_{i-\frac{1}{2},j+\frac{1}{2}} u^n_{i-\frac{1}{2},j+\frac{1}{2}} \Big) \\
&\quad + \Big(\rho_{0,j+1} w^n_{i,j+1} w^n_{i,j+1} - \rho_{0,j} w^n_{i,j} w^n_{i,j} \Big) \Delta x \Big/ (\rho_{0,j+\frac{1}{2}} \Delta z_{j+\frac{1}{2}}) \\
&\quad - w^n_{i,j+\frac{1}{2}} \left(\nabla \cdot \rho_0 \boldsymbol{v} / \rho_{0,j} \right)^n_{i,j+\frac{1}{2}} \Big\}, && (10)
\end{aligned}$$

$$\begin{aligned}
\mathrm{DUVIS}^n_{i+\frac{1}{2},j} &= \frac{1}{(\Delta x)^2} \Big\{ \Big[K^n_{i+1,j} \Big(u^n_{i+\frac{3}{2},j} - u^n_{i+\frac{1}{2},j} \Big) - K^n_{i,j} \Big(u^n_{i+\frac{1}{2},j} - u^n_{i-\frac{1}{2},j} \Big) \Big] \\
&\quad + \frac{(\Delta x)^2}{\rho_{0,j} \Delta z_j} \Big[\rho_{0,j+\frac{1}{2}} K^n_{i+\frac{1}{2},j+\frac{1}{2}} \Big(u^n_{i+\frac{1}{2},j+1} - u^n_{i+\frac{1}{2},j} \Big) \Big/ \Delta z_{j+\frac{1}{2}} - \\
&\quad \rho_{0,j-\frac{1}{2}} K^n_{i+\frac{1}{2},j-\frac{1}{2}} \Big(u^n_{i+\frac{1}{2},j} - u^n_{i+\frac{1}{2},j-1} \Big) \Big/ \Delta z_{j-\frac{1}{2}} \Big] \Big\}, && (11)
\end{aligned}$$

$$
\begin{aligned}
\mathrm{DVVIS}^n_{i+\frac{1}{2},j} &= \frac{1}{(\Delta x)^2}\left\{\left[K^n_{i+1,j}\left(v^n_{i+\frac{3}{2},j}-v^n_{i+\frac{1}{2},j}\right)-K^n_{i,j}\left(v^n_{i+\frac{1}{2},j}-v^n_{i-\frac{1}{2},j}\right)\right]\right. \\
&\quad +\frac{(\Delta x)^2}{\rho_{0,j}\Delta z_j}\left[\rho_{0,j+\frac{1}{2}}K^n_{i+\frac{1}{2},j+\frac{1}{2}}\left(v^n_{i+\frac{1}{2},j+1}-v^n_{i+\frac{1}{2},j}\right)\Big/\Delta z_{j+\frac{1}{2}}-\right. \\
&\quad \left.\left.\rho_{0,j-\frac{1}{2}}K^n_{i+\frac{1}{2},j-\frac{1}{2}}\left(v^n_{i+\frac{1}{2},j}-v^n_{i+\frac{1}{2},j-1}\right)\Big/\Delta z_{j-\frac{1}{2}}\right]\right\}, \qquad (12) \\
\mathrm{DWVIS}^n_{i,j+\frac{1}{2}} &= \frac{1}{(\Delta x)^2}\left\{\left[K^n_{i+\frac{1}{2},j+\frac{1}{2}}\left(w^n_{i+1,j+\frac{1}{2}}-w^n_{i,j+\frac{1}{2}}\right)-K^n_{i-\frac{1}{2},j+\frac{1}{2}}\left(w^n_{i,j+\frac{1}{2}}-w^n_{i-1,j+\frac{1}{2}}\right)\right]\right. \\
&\quad +\frac{(\Delta x)^2}{\rho_{0,j}\Delta z_j}\left[\rho_{0,j+1}K^n_{i,j+1}\left(w^n_{i,j+1}-w^n_{i,j}\right)\Big/\Delta z_{j+1}-\right. \\
&\quad \left.\left.\rho_{0,j}K^n_{i,j}\left(w^n_{i,j+\frac{1}{2}}-w^n_{i,j-\frac{1}{2}}\right)\Big/\Delta z_{j-1}\right]\right\}. \qquad (13)
\end{aligned}
$$

$$
\begin{aligned}
\mathrm{DUNLV}^n_{i+\frac{1}{2},j} &= \left\{\left[\left(u^n_{i+\frac{3}{2},j}-u^n_{i+\frac{1}{2},j}\right)^3-\left(u^n_{i+\frac{1}{2},j}-u^n_{i-\frac{1}{2},j}\right)^3\right]+0.1\left[\left(u^n_{i+\frac{1}{2},j+1}-u^n_{i+\frac{1}{2},j}\right)^3\right.\right. \\
&\quad \left.\left.-\left(u^n_{i+\frac{1}{2},j}-u^n_{i+\frac{1}{2},j-1}\right)^3\right]\right\}\Big/(16.0\cdot 10^3\cdot\rho_{0,j}\Delta z_j/\Delta x), \qquad (14) \\
\mathrm{DVNLV}^n_{i+\frac{1}{2},j} &= \left\{\left[\left(v^n_{i+\frac{3}{2},j}-v^n_{i+\frac{1}{2},j}\right)^3-\left(v^n_{i+\frac{1}{2},j}-v^n_{i-\frac{1}{2},j}\right)^3\right]+0.1\left[\left(v^n_{i+\frac{1}{2},j+1}-v^n_{i+\frac{1}{2},j}\right)^3\right.\right. \\
&\quad \left.\left.-\left(v^n_{i+\frac{1}{2},j}-v^n_{i+\frac{1}{2},j-1}\right)^3\right]\right\}\Big/(16.0\cdot 10^3\cdot\rho_{0,j}\Delta z_j/\Delta x), \qquad (15) \\
\mathrm{DWNLV}^n_{i,j+\frac{1}{2}} &= \left\{\left[\left(w^n_{i+1,j+\frac{1}{2}}-w^n_{i,j+\frac{1}{2}}\right)^3-\left(w^n_{i,j+\frac{1}{2}}-w^n_{i-1,j+\frac{1}{2}}\right)^3\right]\right. \\
&\quad \left.+0.1\left[\left(w^n_{i,j++\frac{3}{2}}-w^n_{i,j+\frac{1}{2}}\right)^3-\left(w^n_{i,j+\frac{1}{2}}-w^n_{i,j-\frac{1}{2}}\right)^3\right]\right\}\Big/ \\
&\quad (16.0\cdot 10^3\cdot\rho_{0,j+\frac{1}{2}}\Delta z_{j+\frac{1}{2}}/\Delta x), \qquad (16) \\
\mathrm{BUOY}^n_{i,j+\frac{1}{2}} &= \frac{g}{\Theta_{0,j+\frac{1}{2}}}\theta^n_{i,j+\frac{1}{2}}, \qquad (17) \\
\mathrm{DUCOLI}^n_{i+\frac{1}{2},j} &= -fv^n_{i+\frac{1}{2},j}, \qquad (18) \\
\mathrm{DVCOLI}^n_{i+\frac{1}{2},j} &= +fu^n_{i+\frac{1}{2},j}. \qquad (19)
\end{aligned}
$$

$$
\begin{aligned}
u^n_{i+\frac{1}{2},j+\frac{1}{2}} = 0.5\left(u^n_{i+\frac{1}{2},j+1}+u^n_{i+\frac{1}{2},j}\right), &\quad u^n_{i,j} = 0.5\left(u^n_{i+\frac{1}{2},j}+u^n_{i-\frac{1}{2},j}\right), \\
v^n_{i+\frac{1}{2},j+\frac{1}{2}} = 0.5\left(v^n_{i+\frac{1}{2},j+1}+v^k_{i+\frac{1}{2},j-1}\right), &\quad v^n_{i,j} = 0.5\left(v^n_{i+\frac{1}{2},j}+v^n_{i-\frac{1}{2},j}\right),
\end{aligned}
$$

$$w^n_{i+\frac{1}{2},j+\frac{1}{2}} = 0.5\left(w^n_{i+1,j+\frac{1}{2}} + w^n_{i,j+\frac{1}{2}}\right), \quad w^n_{i,j} = 0.5\left(w^n_{i,j+\frac{1}{2}} + w^n_{i,j-\frac{1}{2}}\right),$$
$$\theta_{i,j+\frac{1}{2}} = \left(-\frac{1}{16}\theta^n_{i,j+2} + \frac{9}{16}\theta^n_{i,j+1} + \frac{9}{16}\theta^n_{i,j-1} - \frac{1}{16}\theta^n_{i,j-2}\right),$$
$$K^n_{i+\frac{1}{2},j+\frac{1}{2}} = 0.25\left(K^n_{i+1,j+1} + K^n_{i+1,j} + K^n_{i,j+1} + K^n_{i,j}\right),$$

$$\Theta_{0,j+\frac{1}{2}} = 0.5\left(\Theta_{0,j+1} + \Theta_{0,j+1}\right), \quad \rho_{0,j+\frac{1}{2}} = 0.5\left(\rho_{0,j+1} + \rho_{0,j+1}\right),$$

$$(\nabla\cdot\rho_0\boldsymbol{v}/\rho_0)^n_{i+\frac{1}{2},j} = \frac{u^n_{i+1,j} - u^n_{i,j}}{\Delta x} + \frac{\rho_{0,j+\frac{1}{2}}w^n_{i+\frac{1}{2},j+\frac{1}{2}} - \rho_{0,j-\frac{1}{2}}w^n_{i+\frac{1}{2},j-\frac{1}{2}}}{\rho_{0,j}\Delta z_j},$$
$$(\nabla\cdot\rho_0\boldsymbol{v}/\rho_0)^n_{i,j+\frac{1}{2}} = \frac{u^n_{i+\frac{1}{2},j+\frac{1}{2}} - u^n_{i-\frac{1}{2},j+\frac{1}{2}}}{\Delta x} + \frac{\rho_{0,j+1}w^n_{i,j+1} - \rho_{0,j}w^n_{i,j}}{\rho_{0,j+\frac{1}{2}}\Delta z_{j+\frac{1}{2}}}.$$

2.2 Thermodynamics equation

The advection terms $[\mathrm{DTADV}]^n_{i,j}$, $[\mathrm{DTAD0}]^n_{i,j}$ of equation (5) in Part I are evaluated by forth order centered scheme. In time integration, the forward scheme is adapted for the friction terms $[\mathrm{DTDIF}]^N_{i,j}$, $[\mathrm{DTDI0}]^N_{i,j}$, the radiative heating term $Q^N_{rad,i,j}$ and the dissipative heating term $Q^N_{dis,i,j}$. The calculation method of radiative heating term is shown in Section 5.

$$\begin{aligned}\theta^{n+1}_{i,j} &= \theta^N_{i,j} + dt\left\{\frac{\Theta_{0,j}}{T_{0,j}}(Q^N_{rad,i,j} + Q^N_{dis,i,j})\right.\\ &\left.\quad + [\mathrm{DTADV}]^n_{i,j} + [\mathrm{DTAD0}]^n_{i,j} + [\mathrm{DTDIF}]^N_{i,j} + [\mathrm{DTDI0}]^N_{i,j}\right\}\end{aligned} \tag{20}$$

$$Q^N_{dis,i,j} = \frac{C_\epsilon}{lc_p}(\varepsilon^N_{i,j})^{\frac{3}{2}}, \tag{21}$$

$$\begin{aligned}\mathrm{DTADV}^n_{i,j} &= -\left\{\frac{1}{\rho_{0,j}\Delta x}\left[-\frac{1}{24}F\theta^n_{x(i+\frac{3}{2},j)} + \frac{9}{8}F\theta^n_{x(i+\frac{1}{2},j)} - \frac{9}{8}F\theta^n_{x(i-\frac{1}{2},j)} + \frac{1}{24}F\theta^n_{x(i-\frac{3}{2},j)}\right]\right.\\ &\left. + \frac{1}{\rho_{0,j}\Delta z_j}\left[-\frac{1}{24}F\theta^n_{z(i,j+\frac{3}{2})} + \frac{9}{8}F\theta^n_{z(i,j+\frac{1}{2})} - \frac{9}{8}F\theta^n_{z(i,j-\frac{1}{2})} + \frac{1}{24}F\theta^n_{z(i,j-\frac{3}{2})}\right]\right\},\end{aligned} \tag{22}$$
$$F\theta^n_{x(i+\frac{1}{2},j)} = \rho_{0,j}u^n_{i+\frac{1}{2},j}\left(-\frac{1}{16}\theta^n_{i+2,j} + \frac{9}{16}\theta^n_{i+1,j} + \frac{9}{16}\theta^n_{i,j} - \frac{1}{16}\theta^n_{i-1,j}\right),$$

$$F\theta^n_{z(i,j+\frac{1}{2})} = \rho_{0,j+\frac{1}{2}} w^n_{i,j+\frac{1}{2}} \left(-\frac{1}{16}\theta^n_{i,j+2} + \frac{9}{16}\theta^n_{i,j+1} + \frac{9}{16}\theta^n_{i,j} - \frac{1}{16}\theta^n_{i,j-1} \right).$$

$$\begin{aligned}
\mathrm{DTAD0}^n_{i,j} &= -\left\{ \frac{1}{\rho_{0,j}\Delta x}\Theta_{0,j} \left[-\frac{1}{24}F^n_{x(i+\frac{3}{2},j)} + \frac{9}{8}F^n_{x(i+\frac{1}{2},j)} - \frac{9}{8}F^n_{x(i-\frac{1}{2},j)} + \frac{1}{24}F^n_{x(i-\frac{3}{2},j)} \right] \right. \\
&\left. + \frac{1}{\rho_{0,j}\Delta z_j} \left[-\frac{1}{24}F\Theta^n_{0,z(j+\frac{3}{2})} + \frac{9}{8}F\Theta^n_{0,z(j+\frac{1}{2})} - \frac{9}{8}F\Theta^n_{0,z(j-\frac{1}{2})} + \frac{1}{24}F\Theta^n_{0,z(j-\frac{3}{2})} \right] \right\}, \quad (23) \\
F^n_{x(i+\frac{1}{2},j)} &= \rho_{0,j} u^n_{i+\frac{1}{2},j}, \\
F\Theta^n_{0,z(j+\frac{1}{2})} &= \rho_{0,j+\frac{1}{2}} w^n_{i,j+\frac{1}{2}} \left(-\frac{1}{16}\Theta_{0,j+2} + \frac{9}{16}\Theta_{0,j+1} + \frac{9}{16}\Theta_{0,j} - \frac{1}{16}\Theta_{0,j-1} \right).
\end{aligned}$$

$$\begin{aligned}
\mathrm{DTDIF}^n_{i,j} &= \frac{1}{(\Delta x)^2} \left[\tilde{K}^n_{i+\frac{1}{2},j}(\theta^n_{i+1,j} - \theta^n_{i,j}) - \tilde{K}^n_{i-\frac{1}{2},j}(\theta^n_{i,j} - \theta^n_{i-1,j}) \right] + \\
&\quad \frac{1}{\rho_{0,j}\Delta z_j} \left[\rho_{0,j+\frac{1}{2}} \tilde{K}^n_{i,j+\frac{1}{2}} \frac{(\theta^n_{i,j+1} - \theta^n_{i,j})}{\Delta z_{j+\frac{1}{2}}} - \rho_{0,j-\frac{1}{2}} \tilde{K}^n_{i,j-\frac{1}{2}} \frac{(\theta^n_{i,j} - \theta^n_{i,j-1})}{\Delta z_{j-\frac{1}{2}}} \right], \quad (24)
\end{aligned}$$

$$\mathrm{DTDI0}^n_{i,j} = \frac{1}{\rho_{0,j}\Delta z_j} \left[\rho_{0,j+\frac{1}{2}} \tilde{K}^n_{i,j+\frac{1}{2}} \frac{(\Theta^n_{0,j+1} - \Theta^n_{0,j})}{\Delta z_{j+\frac{1}{2}}} - \rho_{0,j-\frac{1}{2}} \tilde{K}^n_{i,j-\frac{1}{2}} \frac{(\Theta^n_{0,j} - \Theta^n_{0,j-1})}{\Delta z_{j-\frac{1}{2}}} \right], \quad (25)$$

$$\tilde{K}^n_{i+\frac{1}{2},j} = 0.5(\tilde{K}^n_{i+1,j} + \tilde{K}^n_{i,j}), \quad \tilde{K}^n_{i,j+\frac{1}{2}} = 0.5(\tilde{K}^n_{i,j+1} + \tilde{K}^n_{i,j}).$$

2.3 Diagonostic equation of pressure function

The diagnostic equation of nodimensional pressure function is solved by using the dimension reduction method. Before making the finite difference equation, equation (9) show in Part I is transformed as follows.

$$\left(\frac{\partial^2}{\partial x^2} + \frac{1}{\rho_0}\frac{\partial}{\partial z}\rho_0\frac{\partial}{\partial z} \right) P = \frac{\partial}{\partial t}\left(\frac{1}{\rho_0}\nabla\cdot\rho_0\boldsymbol{v} \right) + \frac{\partial}{\partial x}\alpha,$$

The finite difference form of the pressure equation can be written in matrix form as follows.

$$\boldsymbol{D_x P} + \boldsymbol{D_z P} = \boldsymbol{S} \tag{26}$$

where $\boldsymbol{P}, \boldsymbol{D_x}, \boldsymbol{D_z}, \boldsymbol{S}$ are matrixes whose elements are finite difference form of following terms.

$$\hat{P}, \quad \frac{\partial^2}{\partial x^2}, \quad \frac{1}{\rho_0}\frac{\partial}{\partial z}\left(\rho_0 \frac{\partial}{\partial z}\right), \quad \frac{\partial}{\partial t}\left(\frac{1}{\rho_0}\nabla\cdot\boldsymbol{v}\right) + \frac{\partial}{\partial x}\alpha,$$

(26)can be rewritten by using the eigenvalue matrix $\boldsymbol{\Lambda}$ and the eigenvector matrix $\boldsymbol{V}$ of $\boldsymbol{D_z}$.

$$\boldsymbol{V D_x H} + \boldsymbol{V \Lambda H} = \boldsymbol{S}.$$

where $\boldsymbol{D_z V} = \boldsymbol{V\Lambda}$ and $\boldsymbol{P} = \boldsymbol{V}\cdot\boldsymbol{H}$. The final form of matrix equation is as follows.

$$(\boldsymbol{D_x} + \Lambda)\,\boldsymbol{H} = \boldsymbol{V}^{-1}\boldsymbol{S}, \tag{27}$$

In calculating elements of matrix $\boldsymbol{D}_z$, the vertical derivative in

$$\frac{1}{\rho_0}\frac{\partial}{\partial z}\rho_0\frac{\partial}{\partial z}\pi$$

are evaluated by the second and forth order centered schemes because the space differencing in the continuity equation is evaluated by the forth order centered scheme while that in the pressure gradient term is evaluated by the second order centered scheme. Therefore $\boldsymbol{D}_z$ is represented as a band matrix whose elements $A_{i,j}$ are given as follows.

$$A_{i,i+2} = -\frac{1}{\rho_0\Delta z_i}\left(\frac{1}{24\Delta z_{i+\frac{3}{2}}}\frac{(\rho_{0,i+2}+\rho_{0,i+1})}{2}\right), \tag{28}$$

$$A_{i,i+1} = \frac{1}{\rho_0\Delta z_i}\left(\frac{1}{24\Delta z_{i+\frac{3}{2}}}\frac{(\rho_{0,i+2}+\rho_{0,i+1})}{2} + \frac{9}{8\Delta m_{i+\frac{1}{2}}}\frac{(\rho_{0,i+1}+\rho_{0,i})}{2}\right), \tag{29}$$

$$A_{i,i} = -\frac{1}{\rho_0\Delta z_i}\left(\frac{9}{8\Delta z m_{i+\frac{1}{2}}}\frac{(\rho_{0,i+1}+\rho_{0,i})}{2} + \frac{9}{8\Delta z_{i-\frac{1}{2}}}\frac{(\rho_{0,i}+\rho_{0,i-1})}{2}\right), \tag{30}$$

$$A_{i+1,i} = \frac{1}{\rho_0\Delta z_i}\left(\frac{1}{24\Delta z_{i-\frac{3}{2}}}\frac{(\rho_{0,i-1}+\rho_{0,i-2})}{2} + \frac{9}{8\Delta z_{i-\frac{1}{2}}}\frac{(\rho_{0,i}+\rho_{0,i-1})}{2}\right), \tag{31}$$

$$A_{i+2,i} = -\frac{1}{\rho_0\Delta z_i}\left(\frac{1}{24\Delta z_{i-\frac{3}{2}}}\frac{(\rho_{0,i-1}+\rho_{0,i-2})}{2}\right). \tag{32}$$

The boundary conditions are $\frac{\partial}{\partial z} = 0$ at the lower and upper boundary.

The horizontal differencing is evaluated by using the Fourier expansion.

$$
\begin{aligned}
\boldsymbol{H} &= \sum_{k=1}^{NX/2-1} [\boldsymbol{H}]_{k_x}, && (33)\\
\boldsymbol{V}^{-1}\boldsymbol{S} &= \sum_{k=1}^{NX/2-1} \left[\boldsymbol{V}^{-1}\boldsymbol{S}\right]_{k_x}, && (34)\\
(\boldsymbol{D_x} + \boldsymbol{\Lambda}) &= \sum_{k_x=1}^{NX/2-1} [\boldsymbol{D_x} + \boldsymbol{\Lambda}]_{k_x} && (35)
\end{aligned}
$$

$$
[\boldsymbol{H}]_{k_x} = [\boldsymbol{D_x} + \boldsymbol{\Lambda}]_{k_x}^{-1} \left[\boldsymbol{V}^{-1}\boldsymbol{S}\right]_{k_x} \tag{36}
$$

2.4 Basic state equations

The basic state pressure ($P_{0,j}$) and density ($\rho_{0,j}$) are calculated by the hydrostatic equation and the equation of state given the basic state temperature $T_{0,j}$.

$$
\ln P_{0,j} = \ln P_{00} - \sum_{j=1}^{j} \frac{g}{RT_{0,j}} \Delta z_j, \tag{37}
$$

$$
\rho_{0,j} = \frac{P_{0,j}}{RT_{0,j}}. \tag{38}
$$

$\Pi_{0,j}$, $\Theta_{0,j}$ are calculated by using $P_{0,j}$, $\rho_{0,j}$ as follows.

$$
\Pi_{0,j} = \left(\frac{P_{0,j}}{P_0 0}\right)^{\kappa}, \tag{39}
$$

$$
\Theta_{0,j} = \frac{T_{0,j}}{\Pi_{0,j}}. \tag{40}
$$

3 Turbulent parameterization

3.1 Subgrid turbulent mixing parameterization

The space differencing in the turbulent kinetic energy equation (equation (10) in Part I) is evaluated by the forth order centered scheme for advection terms and the second order centered scheme for other terms. In time integration, the forward scheme is adapted for the friction terms. Representations of $[\mathrm{DKADV}]^n_{i,j}$ and $[\mathrm{DKDIF}]^N_{i,j}$ are same as those of (22) and (24).

$$\begin{aligned}\varepsilon^{n+1}_{i,j} &= \varepsilon^N_{i,j} + dt\Big\{[\mathrm{DKADV}]^n_{i,j} + [\mathrm{DKDIF}]^N_{i,j} + [\mathrm{DKNLD}]^N_{i,j} \\ &\quad + [\mathrm{DKBP}]^n_{i,j} + [\mathrm{DKSP}]^n_{i,j} - \frac{C_\epsilon}{l}(\varepsilon^N_{i,j})^{\frac{3}{2}}\Big\} \end{aligned} \tag{41}$$

$$\begin{aligned}\mathrm{DKNLD}^n_{i,j} &= \frac{1}{(\Delta x)^2}\left[K^n_{NLD,i+\frac{1}{2},j}(\varepsilon^n_{i+1,j} - \varepsilon^n_{i,j}) - K^n_{NLD,i-\frac{1}{2},j}(\varepsilon^n_{i,j} - \varepsilon^n_{i-1,j})\right] + \frac{1}{\rho_{0,j}\Delta z_j} \\ &\quad \left[\rho_{0,j+\frac{1}{2}}K^n_{NLD,i,j+\frac{1}{2}}\frac{(\varepsilon^n_{i,j+1} - \varepsilon^n_{i,j})}{\Delta z_{j+\frac{1}{2}}} - \rho_{0,j-\frac{1}{2}}K^n_{NLD,i,j-\frac{1}{2}}\frac{(\varepsilon^n_{i,j} - \varepsilon^n_{*i,j-1})}{\Delta z_{j-\frac{1}{2}}}\right]\end{aligned} \tag{42}$$

$$\begin{aligned}K^n_{NLD,i+\frac{1}{2},j} &= \mathrm{MIN}\left[K_{NLD,max}, \quad 0.01\frac{(\Delta x)^2}{\Delta t}\frac{\mathcal{L}(\varepsilon)^n_{i+1,j} + \mathcal{L}(\varepsilon)^n_{i,j}}{2}\right], \\ K^n_{NLD,i,j+\frac{1}{2}} &= \mathrm{MIN}\left[K_{NLD,max}, \quad 0.01\frac{(\Delta x)^2}{\Delta t}\frac{\mathcal{L}(\varepsilon)^n_{i,j+1} + \mathcal{L}(\varepsilon)^n_{i,j}}{2}\right], \\ \mathcal{L}(\varepsilon)^n_{i,j} &= \left(3\left|\varepsilon^n_{i+1,j} + \varepsilon^n_{i-1,j} - 2\varepsilon^n_{i,j}\right| + \left|\varepsilon^n_{i,j+1} + \varepsilon^n_{i,j-1} - 2\varepsilon^n_{i,j}\right|\right)\Big/ 2000, \\ K_{NLD,max} &= 0.2\frac{(\Delta x)^2}{\Delta t}.\end{aligned}$$

$$\begin{aligned}\mathrm{DKBP}^n_{i,j} &= -\frac{g}{\Theta_{0,j}}K^n_{i,j}\frac{1}{\Delta z_j}\left[(\theta_{i,j+\frac{1}{2}} + \Theta_{0,j+\frac{1}{2}}) - (\theta_{i,j-\frac{1}{2}} + \Theta_{0,j-\frac{1}{2}})\right], \quad (43) \\ \mathrm{DKSP}^n_{i,j} &= 2K^n_{i,j}\left[\left(\frac{u^n_{i+\frac{1}{2},j} - u^n_{i-\frac{1}{2},j}}{\Delta x}\right)^2 + \left(\frac{w^n_{i,j+\frac{1}{2}} - w^n_{i,j-\frac{1}{2}}}{\Delta z_j}\right)^2\right]\end{aligned}$$

$$+\frac{2}{3}\varepsilon^n_{i,j}\left[\frac{u^n_{i,j+\frac{1}{2}}-u^n_{i,j-\frac{1}{2}}}{\Delta z_j}+\frac{w^n_{i+\frac{1}{2},j}-w^n_{i-\frac{1}{2},j}}{\Delta x}\right]$$
$$+K^n_{i,j}\left[\frac{u^n_{i,j+\frac{1}{2}}-u^n_{i,j-\frac{1}{2}}}{\Delta z_j}+\frac{w^n_{i+\frac{1}{2},j}-w^n_{i-\frac{1}{2},j}}{\Delta x}\right]^2 \quad (44)$$

$$u^n_{i,j+\frac{1}{2}}=0.5\left(u^n_{i+\frac{1}{2},j+\frac{1}{2}}+u^n_{i-\frac{1}{2},j+\frac{1}{2}}\right),\quad w^n_{i+\frac{1}{2},j}=0.5\left(w^n_{i+\frac{1}{2},j+\frac{1}{2}}+w^n_{i+\frac{1}{2},j-\frac{1}{2}}\right).$$

3.2 Surface flux parameterization

The finite difference form of the surface flux are as follows.

$$F_{u,i} = -\rho_0 C_{D,i}|u_{i,\frac{1}{2}}|u_{i,\frac{1}{2}}, \quad (45)$$
$$F_{\theta,i} = \rho_0 C_{D,i}|u_{i,\frac{1}{2}}|(T_{sfc,i}-T_{i,1}). \quad (46)$$

where

$$C_{D,i}=\begin{cases} C_{Dn}\left(1-\frac{a\mathrm{Ri}_{B,i}}{1+c|\mathrm{Ri}_{B,i}|^{1/2}}\right) & for\quad \mathrm{Ri}_{B,i}<0,\\ C_{Dn}\frac{1}{(1+b\mathrm{Ri}_{B,i})^2} & for\quad \mathrm{Ri}_{B,i}\geq 0,\end{cases} \quad (47)$$

The bulk Richardson number is calculated as follows.

$$\mathrm{Ri}_{B,i}\equiv\frac{gz_1(\Theta_{sfc,i}-\Theta_{i,1})}{\overline{\Theta}_{0,1}u_{i,\frac{1}{2}}}. \quad (48)$$

4 Dust transport

In the advection diffusion equation of dust (equation (20) in Part I), the advection term $[\mathrm{DQADV}]_{i,j}^{n}$ is evaluated by the forth order centered scheme and the vertical advection term associated with the gravitational settling $[\mathrm{DQFALL}]_{i,j}^{N}$ is evaluated by the first order upstream scheme. In time integration, the forward scheme is adapted for the friction terms $[\mathrm{DQDIF}]_{i,j}^{N}$, $[\mathrm{DQNLD}]_{i,j}^{N}$ and the gravitational settling term $[\mathrm{DQFALL}]_{i,j}^{N}$. Representation of $[\mathrm{DQADV}]_{i,j}^{n}$, $[\mathrm{DQDIF}]_{i,j}^{N}$ and $[\mathrm{DQNLD}]_{i,j}^{N}$ are same as those of (22), (24) and (42).

$$q_{i,j}^{n+1} = q_{i,j}^{N} + dt\left\{[\mathrm{DQADV}]_{i,j}^{n} + [\mathrm{DQDIF}]_{i,j}^{N} + [\mathrm{DQFALL}]_{i,j}^{N} + [\mathrm{DQNLD}]_{i,j}^{N}\right\} \tag{49}$$

$$\begin{aligned} \mathrm{DQFALL}_{i,j}^{n} &= -\frac{1}{\rho_{0,j}\Delta z_j}\left\{FQf_{z(i,j+\frac{1}{2})}^{n} - FQf_{z(i,j-\frac{1}{2})}^{n}\right\}, \qquad (50) \\ FQf_{z(i,j-\frac{1}{2})}^{n} &= -\frac{4\rho_d g r_{mod}^2}{18\eta}\left(1 + 2\frac{\lambda_r}{r_{mod}}\frac{p_r}{P_{0,j}}\right)\rho_{0,j} q_{i,j}^{n}, \end{aligned}$$

5 Radiation

The radiative heating rate is calculated from the radiative flux by using the second order centered scheme.

$$Q^n_{rad,i,j} = Q^n_{rad,IR,i,j} + Q^n_{rad,NIR,i,j} + Q^n_{rad,dust,SR,i,j} + Q^n_{rad,dust,IR,i,j}, \tag{51}$$

$$Q_{rad,*,i,j} = -\frac{g}{c_p}\frac{F_{*,net,i,j+\frac{1}{2}} - F_{*,net,i,j-\frac{1}{2}}}{\Delta P_{0,j}}, \tag{52}$$

$$F_{*,net,i,j+\frac{1}{2}} = F^{\uparrow}_{IR,i,j+\frac{1}{2}} - F^{\downarrow}_{IR,i,j+\frac{1}{2}}, \quad \Delta P_{0,j} = P_{0,j+\frac{1}{2}} - P_{0,j-\frac{1}{2}}.$$

The radiative flux is evaluated on the half grid point The subscript m shows grid point in the wave number. In the following sections, the superscript which shows time step is omitted.

5.1 Radiative transfer of atmospheric CO_2

The finite difference form of the infrared radiative flux and the Plank function are represented as follows.

$$\begin{aligned} F^{\uparrow}_{IR,i,j+\frac{1}{2}} &= \sum_m \Delta\nu_m \left\{ \pi B_{\nu_i,T_{sfc,i}} \mathcal{T}_i(0, z_{j+\frac{1}{2}}) \right. \\ &\quad \left. + \sum_{j'=1}^{j} \pi B_{\nu_m,T_{i,j'}} \frac{\mathcal{T}_m(z_{j+\frac{1}{2}}, z_{j'+\frac{1}{2}}) - \mathcal{T}_m(z_{j+\frac{1}{2}}, z_{j'-\frac{1}{2}})}{\Delta z_{j'}} \right\}, \end{aligned} \tag{53}$$

$$F^{\downarrow}_{IR,i,j+\frac{1}{2}} = \sum_m \Delta\nu_m \left\{ \sum_{j'=j}^{J} \pi B_{\nu_m,T_{i,j'}} \frac{\mathcal{T}_m(z_{j+\frac{1}{2}}, z_{j'+\frac{3}{2}}) - \mathcal{T}_m(z_{j+\frac{1}{2}}, z_{j'+\frac{1}{2}})}{\Delta z_{j'}} \right\}, \tag{54}$$

$$B_{\nu_m,T_{i,j}} = \frac{1.19 \times 10^{-8} \nu_m^3}{e^{1.4387\nu_i/T_{i,j}} - 1}, \tag{55}$$

$$T_{i,j} = \Pi_{0,j}(\theta_{i,j} + \Theta_{0,j}) \tag{56}$$

where the averaged transmission function, the equivalent width, and the effective path length are represented as follows.

$$\mathcal{T}_m(z_{j+\frac{1}{2}}, z_{j'+\frac{1}{2}}) = \exp(-W_{m,j,j'}/\Delta\nu_m), \quad W_{m,j,j'} = \frac{S_m u(z_{j+\frac{1}{2}}, z_{j'+\frac{1}{2}})}{\sqrt{1 + S_m u^*(z_{j+\frac{1}{2}}, z_{j'+\frac{1}{2}})/\alpha^*_m}},$$

$$u(z_{j+\frac{1}{2}}, z_{j'+\frac{1}{2}}) = 1.67g|P_{0,j+\frac{1}{2}} - P_{0,j'+\frac{1}{2}}|, \quad \alpha_m^* = \alpha_m \frac{\overline{p}_{jj'}}{p_0},$$

$$P_{0,j+\frac{1}{2}} = 0.5(P_{0,j} + P_{0,j+1}), \quad \overline{p}_{jj'} = \sum_{l=j}^{j'-1} \frac{P_{0,l+1} \cdot 1.67g|P_{0,l+\frac{3}{2}} - P_{0,l+\frac{1}{2}}|}{u(z_{j+\frac{1}{2}}, z_{j'+\frac{1}{2}})},$$

The finite form of the near infrared radiative flux and the effective path length are represented as follows.

$$F^{\downarrow}_{NIR,i,j+\frac{1}{2}} = \sum_m \Delta\nu_m \left\{ S_{\nu_m} \mathcal{T}_i(z_{j+\frac{1}{2}}, z_{J+\frac{3}{2}}) \cos\zeta \right\}. \tag{57}$$

$$u(z_{j+\frac{1}{2}}, z_{J+\frac{3}{2}}) = \frac{1.67g|P_{0,j+\frac{1}{2}} - P_{0,J+\frac{3}{2}}|}{\mathrm{MAX}(\cos\zeta, \epsilon)}.$$

where ϵ is a small parameter to ensure $u = 0$ when $\cos\zeta = 0$.

5.2 Radiative transfer of dust

The finite difference form of the solar radiative transfer equation of dust are represented as follows.

$$\begin{aligned} F^{\uparrow}_{dif,\nu_m,i,j+\frac{1}{2}} &= F^{\uparrow}_{dif,\nu_m,i,j-\frac{1}{2}} - \Delta\tau_{\nu_m,j} \left[\gamma_{1,\nu_m} \frac{F^{\uparrow}_{dif,\nu_m,i,j+\frac{1}{2}} + F^{\uparrow}_{dif,\nu_m,i,j-\frac{1}{2}}}{2} \right. \\ &\quad \left. - \gamma_{2,\nu_m} \frac{F^{\downarrow}_{dif,\nu_m,i,j+\frac{1}{2}} + F^{\downarrow}_{dif,\nu_m,i,j-\frac{1}{2}}}{2} - \gamma_{3,\nu_m} \tilde{\omega}^*_{\nu_m} S_0 e^{-\tau^*_{\nu_m,j+\frac{1}{2}}/\mu_0} \right], \end{aligned} \tag{58}$$

$$\begin{aligned} F^{\downarrow}_{dif,\nu_m,i,j-\frac{1}{2}} &= F^{\downarrow}_{dif,\nu_m,i,j+\frac{1}{2}} + \Delta\tau_{\nu_m,j} \left[\gamma_{2,\nu_m} \frac{F^{\uparrow}_{dif,\nu_m,i,j+\frac{1}{2}} + F^{\uparrow}_{dif,\nu_m,i,j-\frac{1}{2}}}{2} \right. \\ &\quad \left. - \gamma_{1,\nu_m} \frac{F^{\downarrow}_{dif,\nu_m,i,j+\frac{1}{2}} + F^{\downarrow}_{dif,\nu_m,i,j-\frac{1}{2}}}{2} + (1 - \gamma_{3,\nu_m}) \tilde{\omega}^*_{\nu_m} S_0 e^{-\tau^*_{\nu_m,j+\frac{1}{2}}/\mu_0} \right], \end{aligned} \tag{59}$$

$$\Delta\tau_{\nu_m,j} = \frac{\overline{Q}_{e,\nu_m}}{r_{eff}} \frac{3\rho_{0,j} q_{i,j}}{4\pi\rho_s} \Delta z_j, \quad \tau_{\nu_m,j+\frac{1}{2}} = \sum_{j'=j+1}^{J+1} \Delta\tau_{\nu_m,j'}.$$

(58) and (59)can be represented in matrix form as follows.

$$\boldsymbol{F}^{\uparrow}_{dif,\nu_m} = \boldsymbol{A}\boldsymbol{F}^{\uparrow}_{dif,\nu_m} + \boldsymbol{B}\boldsymbol{F}^{\downarrow}_{dif,\nu_m} + \boldsymbol{R}, \tag{60}$$

$$\boldsymbol{F}^{\downarrow}_{dif,\nu_m} = \boldsymbol{C}\boldsymbol{F}^{\uparrow}_{dif,\nu_m} + \boldsymbol{D}\boldsymbol{F}^{\downarrow}_{dif,\nu_m} + \boldsymbol{S}. \tag{61}$$

where $\boldsymbol{F}^{\uparrow}_{dif,\nu_m} = (F^{\uparrow}_{dif,\nu_m,i,\frac{1}{2}}, F^{\uparrow}_{dif,\nu_m,i,\frac{3}{2}}, ..., F^{\uparrow}_{dif,\nu_m,i,J+\frac{1}{2}})^T$ and so on. (60) and (61)are solved by iteration method. The elements of the matrixes in (60) and (61)are represented as follows.

$$A_{jk} = \begin{cases} -\frac{1}{2}\Delta\tau_{\nu_m,j}\gamma_{1,\nu_m} & k = j, j \geq 1 \\ 1 - \frac{1}{2}\Delta\tau_{\nu_m,j}\gamma_{1,\nu_m} & k = j-1, j \geq 1 \\ 0 & \text{others} \end{cases}$$

$$B_{jk} = \begin{cases} A & j = k = 0 \\ \frac{1}{2}\Delta\tau_{\nu_m,j}\gamma_{2,\nu_m} & k = j, j-1, j \geq 1 \\ 0 & \text{others} \end{cases}$$

$$C_{jk} = \begin{cases} \frac{1}{2}\Delta\tau_{\nu_m,j+1}\gamma_{2,\nu_m} & k = j, j+1, j \leq J-1 \\ 0 & \text{others} \end{cases}$$

$$D_{jk} = \begin{cases} -\frac{1}{2}\Delta\tau_{\nu_m,j+1}\gamma_{1,\nu_m} & k = j, j \leq J-1 \\ 1 - \frac{1}{2}\Delta\tau_{\nu_m,j+1}\gamma_{1,\nu_m} & k = j+1 \\ 0 & \text{others} \end{cases}$$

$$R_j = \begin{cases} AS_0 e^{-\tau^*_{\nu_m,\frac{1}{2}}/\mu_0} & j = 0 \\ \Delta\tau_{\nu_m,j}\gamma_{3,\nu_m}\tilde{\omega}^*_{\nu_m} S_0 e^{-\tau^*_{\nu_m,j+\frac{1}{2}}/\mu_0} & j \geq 1 \end{cases}$$

$$S_j = \begin{cases} \Delta\tau_{\nu_m,j+1}(1-\gamma_{3,\nu_m})\tilde{\omega}^*_{\nu_m} S_0 e^{-\tau^*_{\nu_m,j+\frac{3}{2}}/\mu_0} & j \leq J-1 \\ 0 & j = J \end{cases}$$

The finite difference form of the solar radiative transfer equation of dust are represented as follows.

$$\begin{aligned} F^{\uparrow}_{IR,\nu_m,i,j+\frac{1}{2}} &= F^{\uparrow}_{IR,\nu_m,i,j-\frac{1}{2}} - \Delta\tau_{\nu_m,j}\left[\gamma_{1,\nu_m}\frac{F^{\uparrow}_{IR,\nu_m,i,j+\frac{1}{2}} + F^{\uparrow}_{IR,\nu_m,i,j-\frac{1}{2}}}{2}\right. \\ &\quad \left. - \gamma_{2,\nu_m}\frac{F^{\downarrow}_{IR,\nu_m,i,j+\frac{1}{2}} + F^{\downarrow}_{IR,\nu_m,i,j-\frac{1}{2}}}{2} - 2\pi(1-\tilde{\omega}^*_{\nu_m})B_{\nu_m,T_{i,j}}\right], \end{aligned} \tag{62}$$

$$F^{\downarrow}_{IR,\nu_m,i,j-\frac{1}{2}} = F^{\downarrow}_{IR,\nu_m,i,j+\frac{1}{2}} + \Delta\tau_{\nu_m,j}\left[\gamma_{2,\nu_m}\frac{F^{\uparrow}_{IR,\nu_m,i,j+\frac{1}{2}} + F^{\uparrow}_{IR,\nu_m,i,j-\frac{1}{2}}}{2}\right.$$

$$- \gamma_{1,\nu_m} \frac{F^{\downarrow}_{IR,\nu_m,i,j+\frac{1}{2}} + F^{\downarrow}_{IR,\nu_m,i,j-\frac{1}{2}}}{2} + 2\pi(1-\tilde{\omega}^*_{\nu_m}) B_{\nu_m,T_{i,j}} \Bigg] \quad (63)$$

(62) and (63)can be represented in matrix form as follows.

$$\boldsymbol{F}^{\uparrow}_{IR,\nu_m} = \boldsymbol{A}\boldsymbol{F}^{\uparrow}_{IR,\nu_m} + \boldsymbol{B}'\boldsymbol{F}^{\downarrow}_{IR,\nu_m} + \boldsymbol{R}', \quad (64)$$

$$\boldsymbol{F}^{\downarrow}_{IR,\nu_m} = \boldsymbol{C}\boldsymbol{F}^{\uparrow}_{IR,\nu_m} + \boldsymbol{D}\boldsymbol{F}^{\downarrow}_{IR,\nu_m} + \boldsymbol{S}'. \quad (65)$$

where $\boldsymbol{F}^{\uparrow}_{IR,\nu_m} = (F^{\uparrow}_{IR,\nu_m,i,\frac{1}{2}}, F^{\uparrow}_{IR,\nu_m,i,\frac{3}{2}}, ..., F^{\uparrow}_{IR,\nu_m,i,J+\frac{1}{2}})^T$ and so on. (64) and are solved by iteration method. The elements of the matrixes in (64) and are represented as follows.

$$B'_{jk} = \begin{cases} 0 & j=k=1 \\ B_{jk} & \text{others} \end{cases}$$

$$R'_j = \begin{cases} \pi B_{\nu_m,T_{sfc,i}} & j=0 \\ 2\pi\Delta\tau_{\nu_m,j}(1-\tilde{\omega}^*)B_{\nu_m,T_{i,j}} & j \geq 1 \end{cases}$$

$$S'_j = \begin{cases} 2\pi\Delta\tau_{\nu_m,j}(1-\tilde{\omega}^*)B_{\nu_m,T_{i,j}} & j \leq J-1 \\ 0 & i = N \end{cases}$$

6 Ground surface

The time integration of 1D thermal conduction equation of grand surface (equation (55) in Part I is performed by the Crank-Nicolson scheme. The space differencing is evaluated by the second order centered scheme. The grand temperature and depth are evaluated on the grid point and the heat flux is evaluated on the half grid point. The number of vertical grid point is J' and the suffix of the lowest grid point is $j = 1$. The $T_{i,J'}$ is assumed to the surface temperature $T_{sfc,i}$. The finite difference 1D thermal conduction equation is represented as follows.

$$\begin{aligned} \frac{T_{i,j}^{n+1} - T_{i,j}^{n}}{\Delta t} &= \frac{\kappa}{4\Delta z_j}\left(\frac{T_{i,j+1}^{n+1} - T_{i,j}^{n+1}}{\Delta z_{j+1} + \Delta z_j} - \frac{T_{i,j}^{n+1} - T_{i,j-1}^{n+1}}{\Delta z_j + \Delta z_{j-1}}\right. \\ &\quad \left. + \frac{T_{i,j+1}^{n} - T_{i,j+1}^{n}}{\Delta z_{j+1} + \Delta z_j} - \frac{T_{i,j}^{n} - T_{i,j-1}^{n}}{\Delta z_j + \Delta z_{j-1}}\right). \end{aligned} \tag{66}$$

or,

$$\begin{aligned} & -\frac{\kappa\Delta t}{\Delta z_j}\frac{T_{i,j-1}^{n+1}}{\overline{\Delta z}_i} + \left[4 + \frac{\kappa\Delta t}{\Delta z_{i,j}}\left(\frac{1}{\overline{\Delta z}_{j+\frac{1}{2}}} + \frac{1}{\overline{\Delta z}_{j-\frac{1}{2}}}\right)\right] T_{i,j}^{n+1} - \frac{\kappa\Delta t}{\Delta z_j}\frac{T_{i,j+1}^{n+1}}{\overline{\Delta z}_{j+\frac{1}{2}}} \\ = & +\frac{\kappa\Delta t}{\Delta z_j}\frac{T_{i,j-1}^{n}}{\overline{\Delta z}_{j-\frac{1}{2}}} + \left[4 - \frac{\kappa\Delta t}{\Delta z_j}\left(\frac{1}{\overline{\Delta z}_{j+\frac{1}{2}}} + \frac{1}{\overline{\Delta z}_{j-\frac{1}{2}}}\right)\right] T_{i,j}^{n} + \frac{\kappa\Delta t}{\Delta z_j}\frac{T_{i,j+1}^{n+1}}{\overline{\Delta z}_{j+\frac{1}{2}}} \end{aligned} \tag{67}$$

where $\kappa = k_g/\rho_g c_{p,g}$. This equation can be represented in matrix form as follows.

$$\boldsymbol{A} \cdot \boldsymbol{T}^{n+1} = \boldsymbol{B} \cdot \boldsymbol{T}^{n}, \tag{68}$$

where $\boldsymbol{T}^n = (..., T_{i,j}^n, T_{i,j+1}^n, T_{i,j+2}^n, ...)^T$. The elements of $\boldsymbol{A}$, $\boldsymbol{B}$ are represented as follows.

$$A_{jj} = 4 + \frac{\kappa\Delta t}{\Delta z_j}\left(\frac{1}{\overline{\Delta z}_{j+\frac{1}{2}}} + \frac{1}{\overline{\Delta z}_{j-\frac{1}{2}}}\right), A_{j,j+1} = -\frac{\kappa\Delta t}{\Delta z_i}\frac{1}{\overline{\Delta z}_{i+\frac{1}{2}}}, A_{j,j-1} = -\frac{\kappa\Delta t}{\Delta z_i}\frac{1}{\overline{\Delta z}_{i-\frac{1}{2}}},$$

$$B_{jj} = 4 - \frac{\kappa\Delta t}{\Delta z_j}\left(\frac{1}{\overline{\Delta z}_{i+\frac{1}{2}}} + \frac{1}{\overline{\Delta z}_{i-\frac{1}{2}}}\right), B_{j,j+1} = +\frac{\kappa\Delta t}{\Delta z_i}\frac{1}{\overline{\Delta z}_{i+\frac{1}{2}}}, B_{j,j-1} = +\frac{\kappa\Delta t}{\Delta z_i}\frac{1}{\overline{\Delta z}_{i-\frac{1}{2}}},$$

Considering the boundary condition of upper and lower boundaries, (68) is modified as follows.

$$\boldsymbol{A} \cdot \boldsymbol{T}^{n+1} = \boldsymbol{B} \cdot \boldsymbol{T}^{n} + \boldsymbol{S} \tag{69}$$

Therefore, the grand temperature is given by the solution of the following matrix equation.

$$\boldsymbol{T}^{n+1} = \boldsymbol{A}^{-1} \cdot (\boldsymbol{B} \cdot \boldsymbol{T}^{n} + \boldsymbol{S}), \tag{70}$$

where the elements of $\boldsymbol{A}$ and $\boldsymbol{B}$ are modified as follows.

$$\begin{aligned}
A_{11} &= 4 + \frac{\kappa \Delta t}{\Delta z_1}\left(\frac{1}{\overline{\Delta z}_{\frac{3}{2}}}\right), B_{11} = 4 - \frac{\kappa \Delta t}{\Delta z_1}\left(\frac{1}{\overline{\Delta z}_{\frac{3}{2}}}\right), \\
A_{J'J'} &= 4 + \frac{\kappa \Delta t}{\Delta z_{J'}}\left(\frac{1}{\overline{\Delta z}_{J'-\frac{1}{2}}}\right), B_{J'J'} = 4 - \frac{\kappa \Delta t}{\Delta z_{J'}}\left(\frac{1}{\overline{\Delta z}_{J'-\frac{1}{2}}}\right),
\end{aligned}$$

$\boldsymbol{S}$ is a column vector whose dimension is J' are represented as follows.

$$S_j = \begin{cases} \frac{\Delta t}{\rho_g c_{p,g} \Delta z_{J'}}[-F_s(1-A) + F_{IR,net} + H], & j = J' \\ 0, & j \neq J' \end{cases}$$

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