

# 2 重フーリエ変換

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この文書は, 2 重フーリエ変換の基本的な定式化を行う.

## 1 2 重フーリエ変換

$x$  方向に  $[x_{min}, x_{max}]$ ,  $y$  方向に  $[y_{min}, y_{max}]$  の領域での周期関数  $g(x, y)$  のフーリエ変換は

$$s_{k,l} = \frac{D_x}{D_y} \int_{x_{min}}^{x_{max}} \int_{y_{min}}^{y_{max}} g(x, y) \exp\left(-i \frac{2\pi}{D_x} kx - i \frac{2\pi}{D_y} ly\right) dx dy \quad (1)$$

ただし  $D_x = x_{max} - x_{min}$ ,  $D_y = y_{max} - y_{min}$  である. 切断波数  $K, L$  とする逆変換は

$$g(x, y) = \sum_{k=-K}^K \sum_{l=-L}^L s_{k,l} \exp\left(i \frac{2\pi}{D_x} kx + i \frac{2\pi}{D_y} ly\right) \quad (2)$$

$x, y$  それぞれ  $[0, 2\pi]$  の範囲に変換する. すなわち  $x = x_{min} + D_x/(2\pi)\tilde{x}$ ,  $y = y_{min} + D_y/(2\pi)\tilde{y}$  と変換すると正変換, 逆変換はそれぞれ

$$s_{k,l} = \frac{1}{(2\pi)^2} \int_0^{2\pi} \int_0^{2\pi} g(x, y) \exp(-ikx - ily) dx dy, \quad (3)$$

$$g(x, y) = \sum_{k=-K}^K \sum_{l=-L}^L s_{k,l} \exp(ikx + ily). \quad (4)$$

となる. ただし簡単のため  $\tilde{x}$  を  $x, \tilde{y}$  を  $y$  に置き換えた.

$g(x, y)$  が実数であることから  $s_{k,l}$  に制約が加わる. 逆変換の式で波数  $(k, l), (-k, -l)$  成分を書き出すと

$$\begin{aligned}
& s_{k,l} e^{ikx+ily} + s_{-k,-l} e^{-ikx-ily} \\
= & \operatorname{Re}[s_{k,l}] \cos(kx + ly) - \operatorname{Im}[s_{k,l}] \sin(kx + ly) \\
& + i\{\operatorname{Im}[s_{k,l}] \cos(kx + ly) + \operatorname{Re}[s_{k,l}] \sin(kx + ly)\} \\
& + \operatorname{Re}[s_{-k,-l}] \cos(-kx - ly) - \operatorname{Im}[s_{-k,-l}] \sin(-kx - ly) \\
& + i\{\operatorname{Im}[s_{-k,-l}] \cos(-kx - ly) + \operatorname{Re}[s_{-k,-l}] \sin(-kx - ly)\} \\
= & \operatorname{Re}[s_{k,l}] \cos(kx + ly) - \operatorname{Im}[s_{k,l}] \sin(kx + ly) \\
& + i\{\operatorname{Im}[s_{k,l}] \cos(kx + ly) + \operatorname{Re}[s_{k,l}] \sin(kx + ly)\} \\
& + \operatorname{Re}[s_{-k,-l}] \cos(kx + ly) + \operatorname{Im}[s_{-k,-l}] \sin(kx + ly) \\
& + i\{\operatorname{Im}[s_{-k,-l}] \cos(kx + ly) - \operatorname{Re}[s_{-k,-l}] \sin(kx + ly)\} \\
= & \{\operatorname{Re}[s_{k,l}] + \operatorname{Re}[s_{-k,-l}]\} \cos(kx + ly) + \{-\operatorname{Im}[s_{k,l}] + \operatorname{Im}[s_{-k,-l}]\} \sin(kx + ly) \\
& + i\{\operatorname{Im}[s_{k,l}] + \operatorname{Im}[s_{-k,-l}]\} \cos(kx + ly) + i\{\operatorname{Re}[s_{k,l}] - \operatorname{Re}[s_{-k,-l}]\} \sin(kx + ly)
\end{aligned}$$

この虚数部が 0 とならねばならない. したがって

$$\operatorname{Re}[s_{k,l}] = \operatorname{Re}[s_{-k,-l}], \operatorname{Im}[s_{k,l}] = -\operatorname{Im}[s_{-k,-l}], \quad i.e. \quad s_{k,l} = s_{-k,-l}^* \quad (5)$$

である.

## 2 スペクトル計算

流線関数  $\psi(x, y)$  で表される場の全運動エネルギーならびにエンストロフィーを計算し, スペクトル成分で書き表す. 全運動エネルギーは

$$E = \frac{1}{2} \int_{x_{\min}}^{x_{\max}} \int_{y_{\min}}^{y_{\max}} \left[ \left( \frac{\partial \psi}{\partial x} \right)^2 + \left( \frac{\partial \psi}{\partial y} \right)^2 \right] dx dy \quad (6)$$

部分積分すると

$$\begin{aligned}
E &= \int_{y_{\min}}^{y_{\max}} \left[ \psi \frac{\partial \psi}{\partial x} \right]_{x_{\min}}^{x_{\max}} dy + \int_{x_{\min}}^{x_{\max}} \left[ \psi \frac{\partial \psi}{\partial y} \right]_{y_{\min}}^{y_{\max}} dx - \frac{1}{2} \int_{x_{\min}}^{x_{\max}} \int_{y_{\min}}^{y_{\max}} \psi \left[ \frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} \right] dx dy \\
&= -\frac{1}{2} \int_{x_{\min}}^{x_{\max}} \int_{y_{\min}}^{y_{\max}} \psi \left[ \frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} \right] dx dy
\end{aligned}$$

ここで周期境界条件を用いている.  $x, y$  それぞれ  $[0, 2\pi]$  の範囲に変換すべく  $x = x_{\min} + D_x/(2\pi)x^*, y = y_{\min} + D_y/(2\pi)y^*$  とすると,

$$E = -\frac{1}{2} \int_0^{2\pi} \int_0^{2\pi} \psi \left[ \left( \frac{2\pi}{D_x} \right)^2 \frac{\partial^2 \psi}{\partial x^{*2}} + \left( \frac{2\pi}{D_y} \right)^2 \frac{\partial^2 \psi}{\partial y^{*2}} \right] \frac{D_x}{2\pi} \frac{D_y}{2\pi} dx^* dy^*$$

$$= -\frac{D_x D_y}{(2\pi)^2} \frac{1}{2} \int_0^{2\pi} \int_0^{2\pi} \psi \left[ \left( \frac{2\pi}{D_x} \right)^2 \frac{\partial^2 \psi}{\partial x^{*2}} + \left( \frac{2\pi}{D_y} \right)^2 \frac{\partial^2 \psi}{\partial y^{*2}} \right] dx^* dy^*$$

ここで流線関数のスペクトル

$$\psi(x, y) = \sum_{k=-K}^K \sum_{l=-L}^L \tilde{\psi} \exp \left( i \frac{2\pi}{D_x} kx + i \frac{2\pi}{D_y} ly \right) = \sum_{k=-K}^K \sum_{l=-L}^L \tilde{\psi}_{kl} \exp (ikx^* + ily^*) \quad (7)$$

を代入すると,

$$\begin{aligned} E &= \frac{D_x D_y}{(2\pi)^2} \frac{1}{2} \int_0^{2\pi} \int_0^{2\pi} \sum_{k=-K}^K \sum_{l=-L}^L \tilde{\psi}_{k,l} e^{ikx^* + ily^*} \\ &\quad \cdot \sum_{k'=-K}^K \sum_{l'=-L}^L \left[ \left( \frac{2\pi k'}{D_x} \right)^2 + \left( \frac{2\pi l'}{D_y} \right)^2 \right] \tilde{\psi}_{k',l'} e^{ik'x^* + il'y^*} dx^* dy^* \\ &= \frac{D_x D_y}{(2\pi)^2} \frac{1}{2} \sum_{k=-K}^K \sum_{l=-L}^L \sum_{k'=-K}^K \sum_{l'=-L}^L \left[ \left( \frac{2\pi k'}{D_x} \right)^2 + \left( \frac{2\pi l'}{D_y} \right)^2 \right] \tilde{\psi}_{k,l} \tilde{\psi}_{k',l'} \\ &\quad \cdot \int_0^{2\pi} e^{i(k+k')x^*} dx^* \int_0^{2\pi} e^{i(l+l')y^*} dy^* \\ &= \frac{D_x D_y}{(2\pi)^2} \frac{1}{2} \sum_{k=-K}^K \sum_{l=-L}^L \sum_{k'=-K}^K \sum_{l'=-L}^L \left[ \left( \frac{2\pi k'}{D_x} \right)^2 + \left( \frac{2\pi l'}{D_y} \right)^2 \right] \tilde{\psi}_{k,l} \tilde{\psi}_{k',l'} (2\pi)^2 \delta_{k,-k'} \delta_{l,-l'} \\ &= D_x D_y \frac{1}{2} \sum_{k=-K}^K \sum_{l=-L}^L \left[ \left( \frac{2\pi k}{D_x} \right)^2 + \left( \frac{2\pi l}{D_y} \right)^2 \right] \tilde{\psi}_{k,l} \tilde{\psi}_{-k,-l}. \end{aligned}$$

最後に  $\tilde{\psi}_{-k,-l} = \tilde{\psi}_{k,l}^*$  の関係を用いて

$$E = D_x D_y \frac{1}{2} \sum_{k=-K}^K \sum_{l=-L}^L \left[ \left( \frac{2\pi k}{D_x} \right)^2 + \left( \frac{2\pi l}{D_y} \right)^2 \right] |\tilde{\psi}_{k,l}|^2, \quad (8)$$

となる. そこで

$$\frac{1}{2} \left[ \left( \frac{2\pi k}{D_x} \right)^2 + \left( \frac{2\pi l}{D_y} \right)^2 \right] |\tilde{\psi}_{k,l}|^2 \quad (9)$$

をエネルギー・スペクトルの波数  $k, l$  成分と呼ぶことができる.

同様にエンストロフィーは

$$\begin{aligned} Q &= \frac{1}{2} \int_{x_{min}}^{x_{max}} \int_{y_{min}}^{y_{max}} \left[ \frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} \right]^2 dx dy \\ &= \frac{1}{2} \int_0^{2\pi} \int_0^{2\pi} \left[ \left( \frac{2\pi}{D_x} \right)^2 \frac{\partial^2 \psi}{\partial x^{*2}} + \left( \frac{2\pi}{D_y} \right)^2 \frac{\partial^2 \psi}{\partial y^{*2}} \right]^2 \frac{D_x}{2\pi} \frac{D_y}{2\pi} dx^* dy^* \end{aligned}$$

$$\begin{aligned}
&= \frac{D_x D_y}{(2\pi)^2} \frac{1}{2} \int_0^{2\pi} \int_0^{2\pi} \left[ \left( \frac{2\pi}{D_x} \right)^2 \frac{\partial^2 \psi}{\partial x^{*2}} + \left( \frac{2\pi}{D_y} \right)^2 \frac{\partial^2 \psi}{\partial y^{*2}} \right]^2 dx^* dy^* \\
&= \frac{D_x D_y}{(2\pi)^2} \frac{1}{2} \int_0^{2\pi} \int_0^{2\pi} \sum_{k=-K}^K \sum_{l=-L}^L \left[ \left( \frac{2\pi k}{D_x} \right)^2 + \left( \frac{2\pi l}{D_y} \right)^2 \right] \tilde{\psi}_{k,l} e^{ikx^* + il y^*} \\
&\quad \cdot \sum_{k'=-K}^K \sum_{l'=-L}^L \left[ \left( \frac{2\pi k'}{D_x} \right)^2 + \left( \frac{2\pi l'}{D_y} \right)^2 \right] \tilde{\psi}_{k',l'} e^{ik'x^* + il' y^*} dx^* dy^* \\
&= \frac{D_x D_y}{(2\pi)^2} \frac{1}{2} \sum_{k=-K}^K \sum_{l=-L}^L \sum_{k'=-K}^K \sum_{l'=-L}^L \left[ \left( \frac{2\pi k}{D_x} \right)^2 + \left( \frac{2\pi l}{D_y} \right)^2 \right] \left[ \left( \frac{2\pi k'}{D_x} \right)^2 + \left( \frac{2\pi l'}{D_y} \right)^2 \right] \tilde{\psi}_{k,l} \tilde{\psi}_{k',l'} \\
&\quad \cdot \int_0^{2\pi} \int_0^{2\pi} e^{ikx^* + il y^*} e^{ik'x^* + il' y^*} dx^* dy^* \\
&= \frac{D_x D_y}{(2\pi)^2} \frac{1}{2} \sum_{k=-K}^K \sum_{l=-L}^L \sum_{k'=-K}^K \sum_{l'=-L}^L \left[ \left( \frac{2\pi k}{D_x} \right)^2 + \left( \frac{2\pi l}{D_y} \right)^2 \right] \left[ \left( \frac{2\pi k'}{D_x} \right)^2 + \left( \frac{2\pi l'}{D_y} \right)^2 \right] \tilde{\psi}_{k,l} \tilde{\psi}_{k',l'} \\
&\quad \cdot (2\pi)^2 \delta_{k,-k'} \delta_{l,-l'} \\
&= D_x D_y \frac{1}{2} \sum_{k=-K}^K \sum_{l=-L}^L \left[ \left( \frac{2\pi k}{D_x} \right)^2 + \left( \frac{2\pi l}{D_y} \right)^2 \right]^2 \tilde{\psi}_{k,l} \tilde{\psi}_{-k,-l}
\end{aligned}$$

したがって,

$$Q = D_x D_y \frac{1}{2} \sum_{k=-K}^K \sum_{l=-L}^L \left[ \left( \frac{2\pi k}{D_x} \right)^2 + \left( \frac{2\pi l}{D_y} \right)^2 \right]^2 |\tilde{\psi}_{k,l}|^2, \quad (10)$$

となる. そこで

$$Q = \frac{1}{2} \left[ \left( \frac{2\pi k}{D_x} \right)^2 + \left( \frac{2\pi l}{D_y} \right)^2 \right]^2 |\tilde{\psi}_{k,l}|^2, \quad (11)$$

をエンストロフィースペクトルの波数  $k, l$  成分と呼ぶことができる.